Why Do we Call Second Order Logic 'Logic'?

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Abstract

Second order logic has been always considered problematic by modern logicians: so problematic that some of them refuse to call it logic at all. Lindström Theorem sets out a boundary between the "pure logicality" of first order logic and the "mathematicality" of second order logic: is the validity of completeness, compactness and Löwenheim-Skolem Theorem the only qualification to call a formal system "logic"?

I believe second order logic belongs to logic *tout court*, since it fits Tarski's definitions of logical notions as invariants. Since second order logic is categorical, its deductive system is invariant for all the models of a theory formalized in second order logic. By following Tarski's definition this deductive system could then be called logical. It seems to me that this grasps the real sense of what "logical" means and thus it allows all the problems related to the completeness to be overcome.

In addressing the question of whether mathematics is a part of logic, in his *What are Logical Notions?*[1] Tarski gives a perhaps surprising answer: "As you wish!". This is because, as he points out, there are two possible choices, basically reducible to two methods:

> One method is essentially the method of *Principia mathematica*, the method of Whitehead and Russell - the method of types. The second method is the method of people such as Zermelo, von Neumann, and Bernays - the first-order method.

[1] [8], p. 152.

Basically, if we follow the first path the answer is 'yes', while if we follow the second the answer is 'no'. Tarski highlights the importance of the membership relation since it underlies all the set theoretical notions. According to Tarski, within the framework of *Principia Methematica* this relation is invariant, while within the first order method it is not.
Tarski starts with the analysis of the notion of invariance in geometry in order to apply it to logic. According to Tarski only notions that can be considered "invariant under all possible one-to-one transformations of the world onto itself"[2] can be called logic. The result is that we have two different conceptions of logic: the monistic view and the mathematical approach. The point is that both views should be considered "logical". Tarski is not interested in giving an intensional justification of logical systems but an extensional one.
In his forthcoming paper *A Defense of Second Order Logic*, Otavio Bueno concludes, after he reviewing five main criticisms to second order logic, that "this logic is far more robust than the above critics have supposed".[3] In a paper entitled *First order Logic, Second Order Logic and Completeness*, Marcus Rossberg states that "the lack of completeness theorem, despite being an interesting result, cannot be held against the status of second order logic as a proper logic".[4] Also, Jouko Väänänen shows how intertwined the semantics of first and second order logic are and, consequently,

> if we try to analyze why we are not able to decide, e. g. continuum hypothesis, on the basis of ZFC_2, it seems very plausible to develop a theory about what second order quantifiers range over. The first order theory ZFC is exactly such a theory, and it

[2] [8], p. 149
[3] [1], p. 18.
[4] [5], p. 304.

> is indeed the strongest currently available tool for investigating formalizations of second order logic. But this means we are back in the Henkin semantics of second order logic that full second order logic was supposed to avoid.[5]

What do Tarski's idea of what a logical notion is and the issues related to second order logic have in common? The status of second order logic has been regarded (especially from Quine on) as problematic both for technical reasons and for its metalogical entailments. Following the famous Quine thesis that second order logic is "set theory in sheep's clothing", it has been widely maintained that by using second order logic we enter the realm of mathematics and consequently second order logic can no longer be called a proper "logic". On the other hand, the lack of completeness, compactness and Löwenheim-Skolem theorem seems to doom second order to exile from the realm of 'pure' logic, since these features have been always regarded as indispensable in order for any formal system to be called a proper logic.

Various attempts have been made to find a way to handle second order logic. There are, basically, two ways of doing it: either to start from second order logic and "impoverish" it and its properties towards first order, or to start from first order and "enrich" it, thus coming closer and closer to second order. In both cases the creation of "hybrids" has not solved the problem, as the result of Lindström theorem clearly shows.

The key feature of second order logic is its expressive power which is syntetized by its ability to give categorical characterizations of infinite structures. Categoricity should be regarded as important as it is completeness for first order. So why is the validity of completeness, compactness and Löwenheim-Skolem theorem the only qualifications

[5] [9], p. 519

needed in order for a formal system to be called "logic"? They could well be considered only strong *desiderata*. We know that limitative theorems such as incompleteness clearly point out how higher order systems lack of the adequacy between syntax and semantics. Löwenheim theorem, however, shows that even at the first order there is a gap between the two and it states that a theory is unable to "control" its models.

Hence, many ways have been suggested for dealing with second order. I think it is time to move on to a more general level. In the light of the results on second order logic, we should now tackle the problem of which notions and rules are needed to make formalized languages "really" logic. Of course, we need, for instance, a sound deductive system; is it possible to find a way to control deduction in a second order framework to make it sound "enough"? I think this can be done by following the philosophical insight of Tarski's approach to logical notions.

1

The distinction between first and second (or higher) order logic has its origins in the first years of the last century, in particular with the 1915 Löwenheim theorem. Before that, Boole, Schröder, Dedekind, Frege and the first Hilbert, did not need to make such a distinction. The variables freely ranged over objects, properties and relations without any problem. The situation changed with Russell and the Löwenheim theorem which would be extended by Skolem in 1920, when, as Stewart Shapiro notes, "first order languages [became] *de facto* standard in logic".[6]

The distinction between first and second order logic is basically concerns

[6] [6], p. 173.

the “complexity” of the two logics: while, at first order, variables only range over the elements of the domain, at second order they can also range over properties, relations and functions which can exist among the elements of the domain. This is why second order seems so “natural” and essential for a logical formalized language. So far so good. The problem is that when we use second order logic we have to give up some features which have always been regarded as fundamental in order to call a formal system “logic”. In second order logic, in fact, the completeness theorem, the compactness theorem and the Löwenheim-Skolem theorem do not hold.

This is an extremely problematic issue, since, just to mention one consequence, second order lacks of a complete deductive system and thus there is a problematic notion of logical consequence.

On the other hand, the advantages of second order logic in terms of expressive power are undoubted and they are due to the main feature of this type of logic: the ability to provide categorical characterizations of infinite structures. Categoricity is an extremely important property for the axiomatization of infinite structures such as, for instance, arithmetic and analysis, as it can give a clear definition of fundamental concepts and notions which are at the ground level of the axiomatization process.

In this context the semantic aspect of a model-theoretic characterization of second order logic is very important. Let us briefly consider two main semantics for second order: *full* semantics and *Henkin* semantics.

Full semantics is usually associated with second order logic. Its main feature is that, having a fixed domain, the range of first and second order variables is also fixed; therefore the interpretation is unique.

Full semantics is constructed in the usual way. As in first order, a (standard) model is a structure $\langle B, I\rangle$. An assignment is a function that maps an element of the domain B to every first order variable, a subset of the domain B^n to every n-ary relation variable and a function from B^n to B to every n-ary function variable.

Furthermore, the denotative function is obtained by extending the first order function and by adding a new condition: let M=$\langle B, I\rangle$ and s an assignment on M. The denotation of $f^n(t)_n$ for $M, s$is the value of the function $s(f^n)$ for the succession of the elements of B denoted by the elements of $(t)_n$.

Let us extend the satisfaction relation by adding the new cases for second order variables.

Given the relation variable X and a succession of terms $(t)_n$, $M, s \models X(t)_n$ if and only if the succession of the elements of B denoted by the elements of $(t)_n$ is included in $s(X)$.

$M, s \models \forall XA$ if and only if $M, s' \models A$ for any assignments s' such that it coincides with s except on X.

$M, s \models \forall fA$ if and only if $M, s' \models A$ for any assignments s' such that it coincides with s except on f.

As in the case of first order, we say that A is (fully) valid if and only if $M, s \models A$ for every M, s. A is satisfiable if $M, s \models A$ for some M, s. Γ is satisfiable if and only if for every A which belongs to Γ there is a model M, s such that $M, s \models A$. A is a logical consequence of Γ if Γ plus $\{\neg A\}$ is not satisfiable.

Unlike full semantics in which second order variables range over the whole domain (for a certain model), in Henkin semantics relation and function variables only range over fixed subsets of the domain.

A Henkin model of *L2* is a structure $M^H = \langle B, I, D, F\rangle$ where B and I are, as in the standard case, a domain and an interpretation function, while D

and F are successions of relation and function sets. For every n, $D(n)$ is the range of n-ary relation variables and $F(n)$ is the range for n-ary function variables. According to Henkin semantics an assignment is a function mapping an element of the domain to every first order variable, an element of $D(n)$ to every n-ary relation variable and an element of $F(n)$ to every n-ary function variable.

With respect to full semantics new conditions have to be introduced to narrow the range of assignments:

Let $M^H = \langle B, I, D, F \rangle$ a Henkin model and s an assignment on M^H. The denotation of $f^n(t)_n$ for M^H, s is the value of the function $s(f^n)$ for a succession of elements of B denoted by the elements of $(t)_n$.

Given the relation variable X and a succession of terms $(t)_n$, $M^H, s \models X(t)_n$ if the succession of the elements of B denoted by the elements of $(t)_n$ belongs to $s(X)$.

$M^H, s \models \forall X A$ if $M^H, s' \models A$ for every assignment s' such that it coincides with s except on X

$M^H, s \models \forall f A$ if $M^H, s' \models A$ for every assignment s' such that it coincides with s except on f

A formula A is valid if $M^h, s \models A$ for every Henkin model M^H and for every assignments s. AA is satisfiable if $M^h, s \models A$ for some Henkin model M^H and assignment s. Γ is satisfiable if there exists a M^H such that $M^h, s \models A$ for every A in Γ. A is a logical consequence of Γ if, for every Henkin model and every assignment, if $M^H, s \models B$ for every B in Γ, then $M^H, s \models A$.

A Henkin model is called *faithful* if it satisfies the axiom of choice and every instance of the comprehension schema.

This kind of semantics lies between first order semantics and second order semantics. Depending on the direction from which you are looking, it can be regarded as a *widening* of the former or a *narrowing* of the latter.

2

It is natural to question what is the "best" semantics for second order logic. Väänänen gives a harsh answer:

> if second order logic is used in formalizing or axiomatizing mathematics, the choice of semantics is irrelevant: it cannot meaningfully be asked whether one should use Henkin semantics or full semantics. The question arises if we formalize second order logic *after* we have formalized basic mathematical concepts needed for semantics. A choice between the Henkin second order logic and the full second order logic as a primary formalization of mathematics cannot be made; they both come out the same.[7]

In the face of this hard criticism, by using the right semantics second order logic can become a very strong logical tool. We already know that the main feature of this tool is its categoricity.

If models of a theory are isomorphic to each others, they are called categorical, according to the notion of categoricity introduced by the mathematician O. Veblen in 1905. Roughly speaking since these models all have the same structure they thus describe the same reality.

The Löweneheim-Skolem theorem is a limitative theorem which establishes that there is a gap between syntax and semantics also at first order. The form of adequacy between syntax and semantics, which is guaranteed by completeness at first order has always been regarded as a key logical feature. The fact that also at first order this adequacy does not entirely

[7][9], p. 505.

hold also at first order should led us to question its key role. Let us see how this theorem works and how it is linked to categoricity.

The Löweneheim-Skolem theorem states in its more general form that if a theory has an infinite model with cardinality α, then the theory has a model for every cardinal $\geq \alpha$. For a theory to be categorical means, roughly speaking, that all its models are isomorphic. Of course, if a theory has two models of different cardinalities these models cannot be isomorphic. So, at first order Löweneheim-Skolem theorem holds and therefore first order theories are not categorical[8].

Categoricity becomes helpful when we have to give a characterization of the notion of *implicit definition*, one of the primary goals of the logical practice of axiomatization. By itself this would be regarded as a strong point in favor of the logicality of second order. Let us take, for instance, one of the criteria to define implicit definitions used by Shapiro to characterize structures or classes of structures. Shapiro sets two requirements: an *existence condition* such that there exists at least one structure which satisfies the axioms and a *uniqueness condition* such that only one structure can be described (up to isomorphism). The categorical characterization of the most important mathematical structures are basically guaranteed by the very effective interpretation second order model theory provides to the semantics of mathematical languages. The uniqueness condition seems strictly connected to categoricity.

Uniqueness is connected to categoricity and existence is related to coherence. Shapiro writes:

[8]It is possible, also in this case, to weaken or to relativize the notion of categoricity to fit in a first order framework, but in doing so we miss the full strength and the full meaning of what being categorical is.

> A structure is characterized if the axioms are coherent. [...] If Φ is a coherent sentence in a second-order language, then there is a structure that satisfies Φ.[9]

This coherence principle is intimately connected to the more general issues regarding reference and the existence of mathematical entities. What does coherence mean here? It is difficult to identify coherence within a deductive framework since the "logical framework" we are in is of a higher order also for what concerns implicit definition: it seems, therefore, that because of incompleteness of second order it is not possible to link the general notion of coherence with *deductive consistency*. In fact

> Let P be the conjunction of the second-order axioms for Peano arithmetic and let G be a standard Gdel sentence that states the consistency of P. By the incompleteness theorem, $P \wedge \neg G$ is consistent, but it has no models. Indeed, because every model of P is isomorphic to the natural numbers, G is true in all models of P. Clearly, $P \wedge \neg G$ is not a coherent implicit definition of a structure, despite its deductive consistency.[10]

One solution could be to link the notion of coherence to the concept of satisfiability. However, we must be careful of continuously going back-and-forth between different structures. The "power" of categoricity plus the incompletenss can generate a dangerous circolarity. The completeness theorem makes a strong link between provability and validity, and this link binds the relation between categoricity and completeness.

[9] [7], p. 133.
[10] [7], p. 135.

Nonetheless, the use of particular semantics does not fully eliminate the problematic gap between first and second order.

It seems we are still stuck. The problems related to the “staggering” ontology of the entities on which second order quantifiers range over seem to raise more problems. I do not believe that this is entirely the case however. I believe we are at a crossroads. Nevertheless, no matter which path we choose, we will always be within the realm of logic. I shall explain why.

3

At this point I believe a wider and more philosophical reflection is needed. If we want to be provocative we could ask ourselves what it is the actual meaning of the question as to whether or not second order logic to be called logic is. Following an insight of Nelson Goodman we can shift from the question “what is logic” to the question “when is logic”. If this argument is sound, we could perform a metatheorical shifting too, i. e., we can shift from a uniquely deductive orientation to a more semantic and interpretative orientation. In this sense the relationship between language and world gains importance. What about the fact that logic has to be decontextualized and free of ontological entailments? This is true, but in the act of contextualizing the intervention of semantics is unavoidable. In this case second order is a formidable tool since it provides better models for important aspects of the world, like mathematics. If second order does not belong to logic it should at least belong to mathematical logic.

I believe second order logic belongs to logic *tout court*. When Tarski used in *What are logical notions?* the idea that lies behind the Erlagen program in order to give a characterization of what a logical notion is, he did not limit himself to first order. Although not explicitly indicated in Tarski's paper, the idea of invariance could be applied to second (or higher) order predicates and quantifiers.

Thus, if second order notions are invariant they are, according to Tarski, logical notions. However, is second order logic as a whole, a proper "logic"? i.e., is it only necessary that the notions which construct a formal system are regarded as logical for such a system to be called "logic"? I believe that something more is needed, something which can exploit the concept of invariance at a different level.

Basically my argument is this: since second order logic is categorical, its deductive system is invariant for all the models of a theory formalized in second order logic. By following Tarski's definition this deductive system could then be called logical. It seems to me that this grasps the real sense of what "logical" means and thus it allows all the problems related to the completeness of this particular task to be skipped.

Why do I call the deductive system of second order logic "invariant"? Tarski uses the mathematical notion of invariance in which an object is invariant if it remains the same after the action of a transformation (or of a set of transformations). The result of categoricity is that all models are isomorphic to each other. An isomorphism is a map f between sets such that f and its inverse f^{-1} are homomorphism, that is, a mapping which preserves the characteristic structure. That is to say, in more narrative words:

> The word "isomorphism" applies when two complex structures can be mapped onto each other, in such a way that to each part of one structure there is a corresponding part in the other structure, where "corresponding" means that the two parts play similar roles

in their respective structures.[11]

If we have group of models of a theory which are all isomorphic to each other, this means they have the same characteristic structure. In this way we are sure that we can "control" our models. It basically turns out that we are dealing with only one model. We can then be sure that the structural characteristics of this model are invariant. Therefore, the deductive system of a second order theory, being one of its main structural features, is invariant in this sense. Therefore, if we use full semantics for our model theoretic characterization of our second order theory we know that, for instance, satisfiability or logical consequence is invariant within this context.

As was defined above, if we derive a proposition A from a set of propositions Γ this means that A is true in all models which make true all elements of Γ. This does not mean that we have solved the problems of the the effectiveness of second order's deductive system. However, model theoretic semantics is used to establish the metatheoretical results of the system such as validity. It follows that the abundance of valid propositions in second order is uniquely a consequence of the mathematical model used to investigate the relation of logical consequence. In this case the interest in completeness is not so important.

By using this kind of extension of first order model theory we have a relation of logical consequence which could *overgenerate*[12]. Set theoretical propositions gain the level of logical truths. But what do we actually mean when we say that the deductive system is not effective? Shapiro wrote

In first order logic, the completeness theorem shows that deduc-

[11] [3], p. 49.

[12] See [2]

> tive notions are coextensive with their semantic counterparts, but conceptually the notions are distinct. The 'cost' of first order logic is an inability to account for important aspects of mathematics. In particular, semantic notions available in higher order are important in accounting for how mathematical structures are described, how various mathematical structures are interrelated and the presuppositions of various branches of mathematics. In short, there is room for a non-effective consequence relation among the tools of the logician.[13]

In this sense all the characteristics of second order logic are entirely logical. The deductive system associated with any kind of second order logic (i. e., with full or Henkin semantics) is logical notwithstanding completeness; since second order logic is categorical its deductive system is invariant for all the models of a theory formalized in second order logic. By following Tarski's definition it could then be called logical.

This seems to follow an intuitive idea of what logic should be; as Robert Nozick points out:

> I conjecture that logic functions as a filter to weed out data that can safely be ignored. Only what is stable enough and structured enough to pass the tests of logic need to be examined further to detect what particular invariances it exhibits. [...]
>
> The pieces of standard logic function to filter the data of the world, but the filtering could be done differently.[14]

[13][6], p. viii.

[14][4], p. 144

So we are well aware of the results which establish the difference between first and second order. These differences lie in what these theories range over, what their extent is, their power, etc. However, they are both logical theories, constructed by using logical notions. This does not mean they are "all the same"; however, we can now consider them both logical theories. If one wonders what to choose and is concerned since he wants to operate in a purely logical way, we can now naturally answer: "As you wish'!'.

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